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On the flow of certain orientable fluids between two co-axial cones

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The flow between two solid boundary surfaces of revolution and also between two rotating co-axial cones, of certain orientable fluids which are unoriented at rest, has been studied. It is found that for sufficiently low velocity gradients the fluids behave like a Newtonian fluid, while at larger but still very moderate velocity gradients the predicted behaviour is similar to that observed in elastico-viscous fluids. In the latter case it is shown that the flow of this class of fluids in horizontal circles is not possible for the boundary conditions defined above, even when the inertia terms are negligible, and that the normal stresses in the plane perpendicular to the streamlines are not equal.

1. Introduction

Ericksen (1960*a*, *b*) has formulated and developed a theory of anisotropic fluids in which the fluid is characterized by having a single preferred direction at each point, represented by a vector $\mathbf{n}$ of variable magnitude. This direction, which is governed by the fluid motion, may vary with time throughout the fluid. Physically the fluid is pictured as composed of dumb-bell molecules. For the law of change of this vector $\mathbf{n}$, Ericksen introduced the equation (Cartesian tensor notation is employed)

$$\rho \ddot{n}_i = g_i, \tag{1}$$

where dots denote the material derivative. In equation (1) the left-hand expression represents the inertia of the molecules while the vector $\mathbf{g}$ represents the intrinsic forces. Ericksen assumed that:

(*a*) The stress tensor t_{ij} and the vector $\mathbf{g}$ at a particle P at time t are functions of ρ, n_i, $\dot{n}_i$, $v_{i,j}$, being linear in the variables $\dot{n}_i$ and $v_{i,j}$, where ρ is the mass density and v_i is the velocity of the fluid.

(*b*) The forms of t_{ij} and g_i are preserved under all time-dependent proper orthogonal transformations.

(*c*) The structure represented by $\mathbf{n}$ is symmetric with respect to reflexions in the planes parallel and perpendicular to $\mathbf{n}$. He obtained the following explicit expressions for t_{ij} and g_i:

$$t_{ij} = (\alpha_0 + \alpha_1 d_{kk} + \alpha_2 d_{km} n_k n_m + \alpha_3 \hat{n}_k n_k)\,\delta_{ij} + (\alpha_4 + \alpha_5 d_{kk} + \alpha_6 d_{km} n_k n_m + \alpha_7 \hat{n}_k n_k) \times n_i n_j + \alpha_8 d_{ij} + \alpha_9 d_{ik} n_k n_j + \alpha_{10} d_{jk} n_k n_i + \alpha_{11} n_i \hat{n}_j + \alpha_{12} n_j \hat{n}_i, \tag{2}$$

$$g_i = (\beta_0 + \beta_1 d_{kk} + \beta_2 d_{km} n_k n_m + \beta_3 \hat{n}_k n_k)\, n_i + (\alpha_9 - \alpha_{10})\, d_{ij} n_j + (\alpha_{12} - \alpha_{11})\, \hat{n}_i. \tag{3}$$

Here α's and β's are functions of ρ and n^2, δ_{ij} is the Kronecker delta and

$$\left.\begin{aligned} 2d_{ij} &= v_{i,j}+v_{j,i}, \\ \hat{n}_i &= \dot{n}_i-\hat{\omega}_{ij}n_j, \\ 2\hat{\omega}_{ij} &= v_{i,j}-v_{j,i}. \end{aligned}\right\} \tag{4}$$

Ericksen simplified the above equations by making two further assumptions, that the fluid be incompressible and that the inertia of the molecules be negligible. Equations (2) and (3) then reduce to

$$t_{ij} = t_{ji} = -p\delta_{ij}+(\lambda_1+\lambda_2 d_{km}n_k n_m)\,n_i n_j+2\lambda_3 d_{ij}+2\lambda_4(d_{ik}n_k n_j+d_{jk}n_k n_i), \tag{5}$$

$$\dot{n}_i = \hat{\omega}_{ik}n_k+(\mu_1+\mu_2 d_{km}n_k n_m)\,n_i+\mu_3 d_{ij}n_j, \tag{6}$$

where p is an arbitrary pressure while the λ's and μ's are now essentially functions of $n^2 = n_k n_k$, to be determined by experiment, and the overdot denotes the material derivative. Since the fluid is assumed incompressible, the equation of continuity takes the form

$$d_{ii} = 0. \tag{7}$$

The equations of motion, in the absence of body forces, are

$$t_{ij,j} = \rho\dot{v}_i. \tag{8}$$

Solutions of equations (5)–(8) have been given by Ericksen (1960*b*, 1962) for simple shearing flow and homogeneous flows and by Kaloni (1965) for other classical steady flows, which include Poiseuille flow, Couette flow, helical flow, and flow through pipes of arbitrary cross-section. In their analyses of these problems both authors have restricted their attention to fluids for which $\mathbf{n} \to 0$ when the fluid is at rest. The restriction, besides agreeing with the fact that the stress should reduce to hydrostatic pressure when the fluid is at rest or undergoes rigid body motions, as is noted by Green (1964), also has a physical interpretation. In dilute solutions of high polymers at rest, the long-chain (flexible) molecules are supposed to be randomly coiled and, therefore, define no preferred direction. In shear, it is thought that they stretch out and are partially uncoiled so that there is statistically some preferred direction of orientation. In the theory proposed by Ericksen this tendency towards alignment is represented macroscopically by associating with each particle of the fluid a vector $\mathbf{n}$ (of variable magnitude) which represents the direction of orientation and whose magnitude measures the amount of stretch.

The present note is concerned with the analysis of the behaviour of this class of fluids between two solid boundary surfaces of revolution in relative motion about a common vertical axis and also between two rotating coaxial cones, using equations (5) and (6) as constitutive equations. Attention is again focused on fluids for which $\mathbf{n} = 0$ is a stable solution at rest. This condition obviously leads to the question whether $\mathbf{n} = 0$ (a possible solution) is stable when the fluid is in motion or whether there exist other stable solutions for which $\mathbf{n} \neq 0$ during steady motion? In order to explore these possibilities the time-dependent equations (6) for $\mathbf{n}$ have been solved by linearizing them with respect to small perturbations in $\mathbf{n}$ about the assumed steady-state solution. This type of calculation, although not rigorous insofar as complete stability analysis of the solutions is concerned (because the velocity field is not perturbed), is thought to give qualitatively significant information and has the considerable advantage that it reduces the mathematical complexity. In fact, as is noted by Leslie (1964) the correct procedure from the point of view of stability analysis would be to allow perturbations in the velocity vector also and thus make use of a linearized form of equation (8); however, in the situation considered here perturbations of velocity would introduce boundary conditions, a difficulty which is avoided here when only **n** is perturbed; **n** is only subject to initial conditions.

With these reservations, the analysis has been pursued and it is found that at sufficiently low shearing the fluid tends to be unoriented, behaving like a Newtonian fluid. However, at moderate velocity gradients, i.e. when they exceed slightly their critical values to be defined later in the text, orientation occurs and the behaviour shown resembles that predicted for certain other viscoelastic fluid models (cf. Oldroyd 1958). In the latter case our calculations predict that in case of laminar flow there is no simple proportionality between the normal stress functions in the plane perpendicular to streamlines, a fact which is in agreement with the experimental findings of Markovitz & Brown (1962) but seems to contradict those of Roberts (1953).

2. Flow between two rigid boundary surfaces of revolution

We shall first consider the steady flow of an incompressible orientable fluid between two rigid boundaries in relative motion about a common vertical axis and examine the possibility of flow in horizontal circles. We refer to cylindrical polar co-ordinates (r, θ, z), with the z-axis vertically upward and assume the velocity field to be†

$$v_1 = 0, \quad v_2 = r\omega(r,z), \quad v_3 = 0, \tag{9}$$

which satisfies equation (7). The non-vanishing, physical components of the tensors d_{ij} and $\hat{\omega}_{ij}$ then turn out to be

$$\begin{gathered} d_{12} = d_{21} = \tfrac{1}{2}\gamma_1, \quad d_{23} = d_{32} = \tfrac{1}{2}\gamma_2, \\ -\hat{\omega}_{12} = \hat{\omega}_{21} = (\omega + \tfrac{1}{2}\gamma_1), \quad \hat{w}_{23} = -\hat{w}_{32} = \tfrac{1}{2}\gamma_2, \end{gathered} \tag{10}$$

where γ_1 and γ_2 denote $r\,\partial\omega/\partial r$ and $r\,\partial\omega/\partial z$, respectively. On using equation (10), the stress components given by (5) reduce to

$$\left.\begin{aligned} t_{11} &= -p + \Phi n_1^2 + 2\lambda_4 n_1 n_2 \gamma_1, \\ t_{22} &= -p + \Phi n_2^2 + 2\lambda_4 (n_1\gamma_1 + n_3\gamma_2)\, n_2, \\ t_{33} &= -p + \Phi n_3^2 + 2\lambda_4 n_2 n_3 \gamma_2, \\ t_{12} &= \lambda_3\gamma_1 + \Phi n_1 n_2 + \lambda_4[(n^2 - n_3^2)\,\gamma_1 + n_1 n_3 \gamma_2], \\ t_{23} &= \lambda_3\gamma_2 + \Phi n_2 n_3 + \lambda_4[(n^2 - n_1^2)\,\gamma_2 + n_1 n_3 \gamma_1], \\ t_{13} &= \Phi n_1 n_3 + \lambda_4[n_1\gamma_2 + n_3\gamma_1]\, n_2, \end{aligned}\right\} \tag{11}$$

where $$\Phi = [\lambda_1 + \lambda_2(n_1\gamma_1 + n_3\gamma_2)\, n_2].$$

† Here the suffices 1, 2 and 3 refer to the r-, θ- and z-directions.

Assuming $n_i = n_i(r, z, t)$, equations (6) give

$$\left.\begin{aligned} \partial n_1/\partial t &= \tfrac{1}{2}(\mu_3 - 1)\gamma_1 n_2 + [\mu_1 + \mu_2(\gamma_1 n_1 + \gamma_2 n_3) n_2] n_1, \\ \partial n_2/\partial t &= \tfrac{1}{2}(\mu_3 + 1)(\gamma_1 n_1 + \gamma_2 n_3) + [\mu_1 + \mu_2(\gamma_1 n_1 + \gamma_2 n_3) n_2] n_2, \\ \partial n_3/\partial t &= \tfrac{1}{2}(\mu_3 - 1)\gamma_2 n_2 + [\mu_1 + \mu_2(\gamma_1 n_1 + \gamma_2 n_3) n_2] n_3. \end{aligned}\right\} \tag{12}$$

Equations (12) can be satisfied by taking $n_i = 0$ $(i = 1, 2, 3)$, and in that case the theory reduces to that for Newtonian fluids with viscosity $\lambda_3(0)$. However, as has been discussed by Ericksen (1960*b*, 1962), such solutions are, in other situations, of less significance because of their unstable nature than those with $n_i \neq 0$. It therefore becomes natural to inquire whether in this flow, the configuration described by $\mathbf{n} = 0$ is stable or unstable and whether there exist other stable solutions with $n_i \neq 0$? In order to examine these situations we write

$$n_i = N_i + m_i, \tag{13}$$

where the vector $\mathbf{N}$ denotes that value of $\mathbf{n}$ which corresponds to a possible stable steady-state orientation. We now insert the values of n_i from equation (13) into (12), linearize them with respect to m_i, and then determine the stability conditions. The assumption that $\mathbf{N} = 0$ is a stable solution when the fluid is at rest requires that $\mu_1(0) < 0$. Assuming that this holds, a little calculation shows that the solutions of equations (12), when the fluid is in motion, will approach $\mathbf{N} = 0$ in time, only if

$$\gamma < -2\mu_1(0)/\{\mu_3(0)^2 - 1\}^{\frac{1}{2}}, \tag{14}$$

where $\mu_1(0)$ and $\mu_3(0)$ represent their values at $n^2 = 0$, γ is the positive root of the equation

$$\gamma^2 = (\gamma_1^2 + \gamma_2^2) = r^2\{(\partial\omega/\partial r)^2 + (\partial\omega/\partial z)^2\}, \tag{15}$$

and where we have assumed that $|\mu_3(0)| > 1$.

We notice that condition (14) is satisfied for sufficiently low values of γ only. If its value exceeds the critical value γ_c given by

$$\gamma_c^2 = 4\mu_1^2(0)/\{\mu_3^2(0) - 1\}, \tag{16}$$

the associated configuration will be unstable. It therefore follows that if (14) holds, i.e. if $\mathbf{N} = 0$ is stable when the fluid is in motion, the stress components given by (11) will reduce to those for Newtonian fluids. On the other hand, if the condition (14) fails, $\mathbf{N} = 0$ will represent an unstable configuration.

Ignoring now those cases when the fluid behaves like a Newtonian viscous fluid which is quite simple to analyse and also when $\mathbf{N} = 0$ is unstable, which seems to be of little or no interest, we search out solutions which correspond to a stable steady-state $\mathbf{N} \neq 0$. For this, we linearize equations (12) with respect to m_i which then become

$$\partial m_1/\partial t = \tfrac{1}{2}(\bar{\mu}_3 - 1) m_2 \gamma_1 + [\bar{\mu}_1 + \bar{\mu}_2(\gamma_1 N_1 + \gamma_2 N_3) N_2] m_1, \tag{17}$$

$$\partial m_2/\partial t = \tfrac{1}{2}(\bar{\mu}_3 + 1)(\gamma_1 m_1 + \gamma_2 m_2) + [\bar{\mu}_1 + \bar{\mu}_2(\gamma_1 N_1 + \gamma_2 N_3) N_2] m_2, \tag{18}$$

$$\partial m_3/\partial t = \tfrac{1}{2}(\bar{\mu}_3 - 1) m_2 \gamma_2 + [\bar{\mu}_1 + \bar{\mu}_2(\gamma_1 N_1 + \gamma_2 N_3) N_2] m_3, \tag{19}$$

where the bars over μ's denote their values at $n^2 = N^2$. Equations (17)–(19) form a system of linear differential equations, where no term independent of m_i occurs. Hence a general solution of this system of equations is

$$m_i = k_1 \exp\{\xi + \tfrac{1}{2}\gamma(\bar{\mu}_3^2 - 1)^{\frac{1}{2}}\}t + k_2 \exp\{\xi - \tfrac{1}{2}\gamma(\bar{\mu}_3^2 - 1)^{\frac{1}{2}}\}t$$
$$- 2k_3 \exp(\xi t)/\gamma(\bar{\mu}_3^2 - 1)^{\frac{1}{2}}, \quad (20)$$

where k_1, k_2, and k_3 are functions of the space variables only, to be determined by the known initial conditions on the n_i's, and

$$\xi = [\bar{\mu}_1 + \bar{\mu}_2(\gamma_1 N_1 + \gamma_2 N_3) N_2]. \quad (21)$$

It therefore follows that the solutions of equations (17)–(19) will approach stable values for large times, only if

$$\left.\begin{aligned} &[\bar{\mu}_1 + \bar{\mu}_2(\gamma_1 N_1 + \gamma_2 N_3) N_2] < 0, \\ \text{and that} \quad &\tfrac{1}{2}\gamma < -[\bar{\mu}_1 + \bar{\mu}_2(\gamma_1 N_1 + \gamma_2 N_3) N_2]/\{\bar{\mu}_3^2 - 1\}^{\frac{1}{2}}, \end{aligned}\right\} \quad (22)$$

where again it is assumed that $|\bar{\mu}_3| > 1$.

Assuming that the required assumptions are met, i.e. that we have not increased the shear rate so greatly as to violate (22) anywhere, we now solve equations (12) for stable steady values of n_i's by putting $\partial n_i/\partial t = 0$. We may write

$$n_1 = N \sin\psi \cos\delta, \quad n_2 = N\cos\psi, \quad n_3 = N \sin\psi \sin\delta, \quad (23)$$

where $$\tan^2\psi = (\mu_3 - 1)/(\mu_3 + 1) \quad \text{and} \quad \tan\delta = \gamma_2/\gamma_1. \quad (24)$$

On setting the above values of n_i's in the first equation of (12) we arrive at the equation

$$[(\mu_3 - 1)\cot\psi + \mu_2 N^2 \sin 2\psi]\gamma + 2\mu_1 = 0, \quad (25)$$

which, on eliminating ψ from equation (24), is an equation for determining the value of N^2 in terms of the motion.

For real values of the angle ψ, equation (24) requires that $|\mu_3| > 1$, provided $\mathbf{N} \neq 0$. Assuming that this holds, the same equation then gives two possible values of $\tan\psi$, and, therefore, four possible values of the angle ψ. On employing the values of n_i's from equation (23) in the first condition of (22) and combining it with equation (25), we then find that

$$(\mu_3 - 1)\gamma\cot\psi > 0. \quad (26)$$

This condition, for the requirement that $N^2 > 0$, consistent with the theory, then further demands (using equation (25)) that $\mu_3 > 1$. With the implication of this last restriction on μ_3, $\tan\psi$ is then determined uniquely. On employing the value of ψ, so determined, in equation (25), we thus get the real values of N^2 in terms of the motion. We assume that this equation is soluble.

To summarize what has been said above, we note from equation (14) that the vector $\mathbf{N}$ tends to remain zero in slow flows when the velocity gradients are sufficiently small. This means that molecules tend statistically to take spherical shapes and remain coiled until a critical shear rate, defined by (16), is reached. At this point they begin to stretch out and are partially uncoiled until the conditions (22) are violated. In this range the vector $\mathbf{N}$ takes the real values given by (23). These are, however, not unique, for if **N** is a solution, so also is $-\mathbf{N}$, since the whole theory is invariant under the substitution of $-\mathbf{n}$ for $\mathbf{n}$. [It is possible that there may exist other stable solutions: the solutions which have been labelled unstable here may in fact be stable. A complete and rigorous analysis would require consideration of the velocity perturbations also, as was done by Leslie (1964).]

So far we have not considered the flow as a whole, but only viewed the behaviour of **n** from a spatially localized point of view. We now develop the analysis further, assuming that we have exceeded the critical shear rate given by (16), but not to the point of violating (22). The stress components on using (23) then turn out to be

$$\left.\begin{aligned} t_{11} &= -p+(\lambda_1 N^2 \sin^2\psi + A\gamma)\cos^2\delta, \\ t_{22} &= -p+(\lambda_1 N^2 \cos^2\psi + B\gamma), \\ t_{33} &= -p+(\lambda_1 N^2 \sin^2\psi + A\gamma)\sin^2\delta, \\ t_{12} &= (C+D\gamma)\cos\delta, \\ t_{23} &= (C+D\gamma)\sin\delta, \\ t_{13} &= (\lambda_1 N^2 \sin^2\psi + A\gamma)\sin\delta\cos\delta, \end{aligned}\right\} \tag{27}$$

where

$$\left.\begin{aligned} &2A = N^2 \sin 2\psi[\lambda_2 N^2 \sin^2\psi + 2\lambda_4],\ 2B = N^2 \sin 2\psi\,[\lambda_2 N^2 \cos^2\psi + 2\lambda_4], \\ &2C = \lambda_1 N^2 \sin 2\psi,\ 4D = 4(\lambda_3 + \lambda_4 N^2) + \lambda_2 N^4 \sin^2 2\psi \end{aligned}\right\} \tag{28}$$

and N^2 is given by equation (25).

It is convenient to introduce three material functions defined by

$$(C+D\gamma) \equiv \alpha(\gamma), \quad (\gamma_1 N^2 \sin^2\psi + A\gamma) \equiv \beta_1(\gamma), \quad (\lambda_1 N^2 \cos^2\psi + B\gamma) \equiv \beta_2(\gamma), \tag{29}$$

where $\alpha(\gamma)$ and $\beta(\gamma)$'s are functions of γ only ($\alpha(\gamma)$ being an odd and the $\beta(\gamma)$'s being even functions) as is evident from the above equations, and whose properties have been discussed earlier (Kaloni 1965). $\alpha(\gamma)/\gamma$ is the 'shear dependent viscosity' and $\beta_1(\gamma)$ and $\beta_2(\gamma)$ are the 'normal stress difference functions'.

On employing the above values of the stress components, the equations of motion (8), transformed into cylindrical polar form, in which the stress components are functions of r and z only, take the form

$$\frac{\partial}{\partial r}\left[\frac{\beta_1(\gamma)}{\gamma^2}\gamma_1^2\right]+\frac{\partial}{\partial z}\left[\frac{\beta_1(\gamma)}{\gamma^2}\gamma_1\gamma_2\right]+\frac{1}{r}\left[\frac{\beta_1(\gamma)\gamma_1^2}{\gamma^2}-\beta_2(\gamma)\right] = \frac{\partial p}{\partial r}-\rho r\omega^2, \tag{30}$$

$$\frac{\partial}{\partial r}\left[\frac{\alpha(\gamma)}{\gamma}\gamma_1\right]+\frac{2}{r}\left[\frac{\alpha(\gamma)}{\gamma}\gamma_1\right]+\frac{\partial}{\partial z}\left[\frac{\alpha(\gamma)}{\gamma}\gamma_2\right] = 0, \tag{31}$$

$$\frac{\partial}{\partial r}\left[\frac{\beta_1(\gamma)}{\gamma^2}\gamma_1\gamma_2\right]+\frac{\partial}{\partial z}\left[\frac{\beta_1(\gamma)}{\gamma^2}\gamma_2^2\right]+\frac{1}{r}\left[\frac{\beta_1(\gamma)}{\gamma^2}\gamma_1\gamma_2\right] = \frac{\partial p}{\partial z}+\rho g, \tag{32}$$

where $\alpha(\gamma)$ and the $\beta(\gamma)$'s have the values given by (29).

Equation (31) is the differential equation governing $\omega(r, z)$ along with the known prescribed boundary conditions. It is a similar differential equation to that obtained for the velocity of unrectilinear flow of the class of fluids which are characterized by a single variable coefficient of viscosity $\eta(\gamma) = \alpha(\gamma)/\gamma$. In a particular case when $\alpha(\gamma)/\gamma = \text{const.}$, i.e. when the viscosity of the fluid is constant, this equation reduces to the corresponding equation for a Newtonian fluid. Equations (30) and (32), however, lead to an additional condition on the velocity $r\omega(r, z)$, which must be satisfied if steady motion in horizontal circles of this class of fluids is to be possible. Eliminating p between these equations, we obtain the condition for consistency as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\partial^2}{\partial z^2}-\frac{1}{r^2}\right)\left(\frac{\beta_1(\gamma)}{\gamma^2}\gamma_1\gamma_2\right)+\frac{\partial^2}{\partial r\,\partial z}\left[\frac{\beta_1(\gamma)}{\gamma^2}(\gamma_2^2-\gamma_1^2)\right] + \frac{1}{r}\frac{\partial}{\partial z}\left[\frac{\beta_2(\gamma)\,\gamma^2-\beta_1(\gamma)\,\gamma_1^2}{\gamma^2}\right] = \rho r\frac{\partial w^2}{\partial z}. \tag{33}$$

Only in the very special case, when ω is a function of r only is the condition (33) automatically satisfied; but, in general, it imposes a restriction on the steady flows in horizontal circles which are physically possible. If the fluid is assumed to be so highly viscous that inertia terms in the equation of motion are negligible, then condition (33), in general, requires that $\beta_1 = \beta_2 = 0$. It therefore follows that for fluids in which the material functions λ_1, λ_2 and λ_4 all vanish, no restrictions are imposed on the velocity pattern, with inertia terms omitted. These conclusions are somewhat similar to those noted by Oldroyd (1958) in the case of certain idealized elastico-viscous fluids. In fact, it is interesting to point out that condition (33) goes over to equation (44) of Oldroyd's paper if the material functions $\beta_1(\gamma)$ and $\beta_2(\gamma)$ used here have the following relationship with the various constants involved in the constitutive equation of Oldroyd:

$$\left.\begin{aligned}\beta_1(\gamma) &= -[(\lambda_1-\mu_1)\,F(\gamma)-(\lambda_2-\mu_2)\,\eta_0]\,\gamma^2,\\ \beta_2(\gamma) &= [(\lambda_1+\mu_1)\,F(\gamma)-(\lambda_2+\mu_2)\,\eta_0]\,\gamma^2.\end{aligned}\right\} \tag{34}$$

3. Flow between two co-axial cones

We now consider the motion of incompressible orientable fluids contained between two co-axial cones of common vertex and vertical axis, which are rotating about their common axis with velocities Ω_1 and Ω_2, respectively. We assume the motion of the fluid particles to be steady and in circles confined to the planes perpendicular to the axis of cones. Although the present case is a special case of §2, we feel it convenient to treat it independently. Referred to spherical polar co-ordinate system (r, θ, ϕ) we assume a velocity distribution, with physical components, of the form†

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = rw(\theta)\sin\theta. \tag{35}$$

The boundary conditions then take the form

$$\omega = \Omega_1 \quad \text{on} \quad \theta = \theta_1, \qquad \omega = \Omega_2, \quad \text{on} \quad \theta = \theta_2, \tag{36}$$

where θ_1 and θ_2 are the semi-vertical angles of the inner and outer cones respectively.

The physical, non-vanishing, components of the tensors d_{ij} and $\hat{w}_{ij}$, in this case reduce to

$$d_{23} = d_{32} = \tfrac{1}{2}\gamma, \quad \hat{w}_{32} = -\hat{w}_{23} = \tfrac{1}{2}\gamma, \tag{37}$$

† The suffices 1, 2 and 3 now refer to the r-, θ- and ϕ-directions.

where now
$$\gamma = (d\omega/d\theta)\sin\theta. \tag{38}$$
On using (37), the physical components of the stress tensor are given by
$$\left.\begin{aligned} t_{11} &= -p+\Phi n_1^2, \\ t_{22} &= -p+\Phi n_2^2+2\lambda_4\gamma n_2 n_3, \\ t_{33} &= -p+\Phi n_3^2+2\lambda_4\gamma n_2 n_3, \\ t_{12} &= \Phi n_1 n_2+\lambda_4\gamma n_1 n_3, \\ t_{13} &= \Phi n_1 n_3+\lambda_4\gamma n_1 n_2, \\ t_{23} &= \Phi n_2 n_3+\gamma[\lambda_4(n^2-n_1^2)+\lambda_3], \end{aligned}\right\} \tag{39}$$
where now
$$\Phi = (\lambda_1+\lambda_2\gamma n_2 n_3).$$
Assuming $n_i = n_i(\theta, t)$, the equations (6) become
$$\left.\begin{aligned} \partial n_1/\partial t &= (\mu_1+\mu_2\gamma n_2 n_3)\, n_1, \\ \partial n_2/\partial t &= \tfrac{1}{2}(\mu_3-1)\,\gamma n_3+(\mu_1+\mu_2\gamma n_2 n_3)\, n_2, \\ \partial n_3/\partial t &= \tfrac{1}{2}(\mu_3+1)\,\gamma n_2+(\mu_1+\mu_2\gamma n_2 n_3)\, n_3. \end{aligned}\right\} \tag{40}$$
The stability criteria for these equations can be analysed on the same lines as given in the previous section. We find that conditions (14) to (16) still hold true except that now γ is given by equation (38). Also the stability conditions for $\mathbf{N} \neq 0$, assuming $|\bar{\mu}_3| > 1$, now become
$$\left.\begin{aligned} &(\bar{\mu}_1+\bar{\mu}_2\gamma n_2 n_3) < 0, \\ &\gamma < -2[\bar{\mu}_1+\bar{\mu}_2\gamma n_2 n_3]/\{\bar{\mu}_3^2-1\}^{\frac{1}{2}}, \end{aligned}\right\} \tag{41}$$
with γ now given by (38). The values of the n_i's for steady-state flow, when $\mathbf{N} \neq 0$ represents a stable configuration, turn out to be
$$n_1 = 0, \quad n_2 = N\sin\psi, \quad n_3 = N\cos\psi, \tag{42}$$
where ψ is given by (24). On setting these values of n_i's in the second equation of (40), we again arrive at equation (25), with γ given by (38), which determines N^2 in terms of the motion.

The stress components on using (42) become
$$\left.\begin{aligned} &t_{11} = -p, \quad t_{22} = -p+\beta_1(\gamma), \quad t_{33} = -p+\beta_2(\gamma), \\ &t_{23} = \alpha(\gamma), \quad t_{12} = t_{13} = 0, \end{aligned}\right\} \tag{43}$$
where $\alpha(\gamma)$ and the $\beta(\gamma)$'s are the material functions introduced earlier and γ is given by (38).

It is clear from the above equations that the stresses are functions of θ only. Hence the dynamical equation (5), on transforming to spherical polar form and on employing the above values of the stress components, becomes
$$-[\beta_1(\gamma)+\beta_2(\gamma)]/r = \partial p/\partial r - \rho r\omega^2 \sin^2\theta, \tag{44}$$
$$\frac{1}{r}\frac{d\beta_1(\gamma)}{d\theta}+\frac{1}{r}[\beta_1(\gamma)-\beta_2(\gamma)]\cot\theta = \frac{1}{r}\frac{\partial p}{\partial\theta} - \rho r\omega^2\sin\theta\cos\theta, \tag{45}$$
$$\frac{1}{r}\frac{d\alpha(\gamma)}{d\theta}+\frac{2\alpha(\gamma)}{r}\cot\theta = 0. \tag{46}$$

Equation (46), on integration, gives

$$\alpha(\gamma) = \kappa \operatorname{cosec}^2 \theta, \tag{47}$$

where κ is a constant to be determined from boundary conditions. As in the analysis of the previous section, the first two equations, (44) and (45), lead to an additional condition on the function $\omega(\theta)$, indicating that it would not be possible to maintain the velocity distribution of the form (35). Eliminating p, between these equations we obtain the condition for the steady motion in horizontal circles as

$$\rho r^2 \omega (d\omega/d\theta) \sin^2 \theta = d[\beta_1(\gamma) + \beta_2(\gamma)]/d\theta. \tag{48}$$

Only when condition (48) is satisfied, does equation (47) along with the known boundary conditions determine a possible steady velocity distribution. We note that this condition is satisfied only when each side of (48) is separately equal to zero, i.e. when the inertia terms in the equations of motion are zero and when

$$[\beta_1(\gamma) + \beta_2(\gamma)] = \text{const.} \tag{49}$$

It, therefore, follows that for a velocity distribution of the form (35) to hold, a necessary condition is that the inertial terms in the equation of motion should be negligible. Assuming that this holds true, i.e. if the liquid be highly viscous, the second condition, (49), then requires that both γ and n^2 must be constant. This restriction, however, contradicts (47) as well as the basic assumptions of the theory. We therefore conclude that the flow of such fluids in horizontal circles, under general conditions, is not possible in a wide gap between cones, even if the inertial effects are negligible. This conclusion is identical with that observed by Oldroyd (1958), Ericksen (1960*c*) and Bhatnagar & Rathna (1963) for various isotropic theories of visco-elastic fluids.

4. Comparison with experimental results

It is of some interest to compare the above results with the experimental observations of Roberts (1953) and Markovitz & Brown (1962), which have been obtained by shearing some real visco-elastic fluids between a horizontal flat plate and a cone of semi-vertical angle very near to $\frac{1}{2}\pi$. If the semi-vertical angles of the two cones are sufficiently near to $\frac{1}{2}\pi$, then in equation (46) the second term vanishes and we get

$$\alpha(\gamma) = \text{const.}, \tag{50}$$

which requires that $\gamma = \gamma_0$, a constant. Roberts, by shearing a number of colloidal and polymer solutions between a horizontal flat plate and a cone of semi-vertical angle very near to $\frac{1}{2}\pi$, concluded that the distribution of the normal stresses was equivalent to an extra-tension along the streamlines, the normal stresses in any plane perpendicular to the streamlines being equal. In terms of the variables defined above, if the above conclusions are thought to be correct, then we must have

$$t_{\theta\theta} = t_{rr}$$

and

$$t_{rr} < t_{\phi\phi}, \quad t_{\theta\theta} < t_{\phi\phi}. \tag{51}$$

An inspection of the stress distribution (43), reveals that in the present case

$$t_{rr} \neq t_{\theta\theta} \neq t_{\phi\phi}, \tag{52}$$

which is in contradiction with the first condition given by (51). The second condition of (51), however, can be satisfied if it is assumed that all λ's are positive definite functions of n^2 and therefore of γ, and if the angle ψ, which the preferred direction makes with the streamlines, lies between 0° and 45°. In their experimental predictions, Markovitz & Brown (1962) have, however, obtained results of the form (52) which seem to hold for this class of fluids also. Very recently Adams & Lodge (1964) have made assumptions similar to (52) for investigating the rheological properties of concentrated polymer solutions. These results, therefore, lead us to conclude that in visco-elastic fluids, relations of the type (52) are acceptable but not of the form (51).

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